

# AN ATTEMPT TO CONSTRUCT DYNAMICAL EVOLUTION IN QUANTUM FIELD THEORY

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**ABSTRACT.** If we develop into perturbation series the evolution operator of the Heisenberg equation in the infinite dimensional Weyl algebra, say, for the  $\varphi^4$  model of field theory, then the arising integrals almost coincide with the usual Feynman diagram integrals. This fact leads to some mathematical definitions which, as it seemed to the author, defined dynamical evolution in quantum field theory in a mathematically rigorous way using the Weyl algebra. In fact the constructions of the paper are well defined in perturbation theory only in one-loop (quasiclassical) approximation. A variation of the construction is related with the Bogolyubov  $S$ -matrix  $S(g)$ .

This paper exposes an attempt to construct dynamical evolution in quantum field theory. The paper is based on the developments of the previous papers [1–5]. In particular, in [5] a complete mathematical theory of free boson field is exposed. In the present paper we announce a generalization of results of [5] to interacting fields. This paper freely uses the results and notations of [5].

## 1. DEFINITION OF AN ATTEMPT OF DYNAMICAL EVOLUTION

**Definition.** An attempt of dynamical evolution (DE) in quantum field theory on the space-time  $\mathbb{R}^{n+1}$  with coordinates  $x^0 = t, x^1, \dots, x^n$  and with boson fields  $u^1, \dots, u^m$  is the following object. Consider any parameterized space-like surface  $\mathcal{C}$  in  $\mathbb{R}^{n+1}$ :

$$(1) \quad x^j = x^j(s), \quad 0 \leq j \leq n; \quad s = (s^1, \dots, s^n).$$

With this surface one can associate the infinite dimensional Weyl algebra (see its definition in [5], Subsect. 2.3) of functionals  $\Phi(u^i(s), p^i(s))$  on the phase space, where  $p^i$  are the variables conjugate to  $u^i$  (see [5]). Denote this Weyl algebra by  $W_{\mathcal{C}}$ . A DE associates with each pair of parameterized space-like surfaces  $\mathcal{C}_1, \mathcal{C}_2$  an isomorphism of algebras  $W_{\mathcal{C}_1} \rightarrow W_{\mathcal{C}_2}$ . These isomorphisms should constitute an infinitely differentiable family with respect to varying pair  $\mathcal{C}_1, \mathcal{C}_2$  in the natural

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Partially supported by the grant RFFI N 04-01-00640.

sense. If the surfaces  $\mathcal{C}_1$  and  $\mathcal{C}_2$  present two parameterizations of one and the same space-like surface, then the isomorphism  $W_{\mathcal{C}_1} \rightarrow W_{\mathcal{C}_2}$  should coincide with the natural action of a change of variables  $s$  on functionals  $\Phi(u^i(s), p^i(s))$ . (The variables  $u^i(s)$  are transformed like functions, and  $p^i(s)$  are transformed like densities.) The isomorphisms should also satisfy the associativity condition: for any triple  $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$  the isomorphism  $W_{\mathcal{C}_1} \rightarrow W_{\mathcal{C}_3}$  should coincide with the composition of isomorphisms  $W_{\mathcal{C}_1} \rightarrow W_{\mathcal{C}_2}$  and  $W_{\mathcal{C}_2} \rightarrow W_{\mathcal{C}_3}$ . Finally, this family of isomorphisms should be invariant with respect to the given action of the symmetry group (the Poincare group, or the gauge group) on the space of variables  $x, u$ .

An *attempt of quantization* of a classical field theory given by a variational principle ([5], formula (2)) is a DE depending on the parameter  $\hbar$  (which appears in the definition of multiplication in the Weyl algebra) such that its classical limit as  $\hbar \rightarrow 0$  coincides with the family of isomorphisms of Poisson algebras of functionals  $\Phi(u^i(s), p^i(s))$  on the phase spaces of space-like surfaces  $\mathcal{C}_1, \mathcal{C}_2$ ; these isomorphisms of Poisson algebras are given by the evolution operators of the generalized canonical Hamilton equations ([5], eq. (16)).

## 2. PERTURBATION SERIES EXPANSION

In this Section we will mainly consider so-called attempt of restricted dynamical evolution (RDE) involving only flat space-like surfaces  $t = \text{const}$ . By definition, a RDE is a one-parametric group of automorphisms of the Weyl algebra  $W$  of functionals  $\Phi(u^i(\mathbf{x}), p^i(\mathbf{x}))$ ,  $\mathbf{x} = (x_1, \dots, x_n)$ . Any Poincare invariant DE yields a RDE after restriction to flat space-like surfaces of constant time.

A basic example of DE is the free scalar field, see [5], §3. The corresponding RDE is given by the Heisenberg equation (in the notations of loc. cit.)

$$(2) \quad i\hbar \frac{\partial \Phi}{\partial t} = [H_0, \Phi],$$

where  $H_0$  is the Hamiltonian of a free field. Since  $H_0$  is quadratic, we have

$$(3) \quad \frac{1}{i\hbar} [H_0, \Phi] = \{\Phi, H_0\},$$

hence the evolution operator  $A_0(t) : W \rightarrow W$  at the time  $t$  is given by the linear symplectic change of variables given by the evolution operator of the Klein–Gordon equation ([5], eq. (42)). In [5] this evolution operator  $A_0(t)$  was symbolically denoted by conjugation by  $\exp tH_0/(i\hbar)$  in the Weyl algebra. However, if one attempts to compute

$\exp tH_0/(ih)$ , then one immediately realizes that this exponent does not exist in the Weyl algebra, since already  $H_0 * H_0$  does not exist. We conjecture that for  $t \neq 0$ ,  $A_0(t)$  is not an inner automorphism of  $W$  (i. e. not a conjugation by an element of  $W$ ).

Consider now the  $\varphi^4$  model as a typical illustration of the perturbation theory method. The classical  $\varphi^4$  theory is given by the variational principle ([5], formulas (2),(13)) with the potential term  $V = \frac{m^2}{2}u^2 + \frac{g}{4!}u^4$ , where  $g$  is the coupling constant. Consider the Heisenberg equation in the Weyl algebra

$$(4) \quad ih \frac{\partial \Phi}{\partial t} = [H, \Phi],$$

where  $H$  is the Hamiltonian of the  $\varphi^4$  theory. If we apply usual perturbation expansion to the evolution of this equation from the time  $t_0$  to the time  $t_1$ , i. e. if we formally develop into series the symbolical expression

$$(5) \quad \exp \frac{t_1 - t_0}{ih} H * \exp \frac{t_0 - t_1}{ih} H_0,$$

then a computation in the Weyl algebra shows that the perturbation series is given by the usual Feynman diagram integrals in configuration space, with two differences: 1) instead of the usual Feynman propagator  $1/(p^2 - m^2 + i\varepsilon)$  we have the propagator  $\mathcal{PV} 1/(p^2 - m^2)$ , where  $\mathcal{PV}$  denotes the Cauchy principal value; 2) integration goes not over the whole space-time  $\mathbb{R}^{n+1}$  but over the domain between the planes  $t = t_0$  and  $t = t_1$ . Hence these integrals are divergent. Let us call them modified Feynman integrals.

Now we make the key assumption which seems reasonable. Assume that one can naturally renormalize the modified Feynman integrals. Denote by  $U(t)$ ,  $t = t_1 - t_0$ , the renormalized perturbation series, and by  $A(t)$  the corresponding operator in the Weyl algebra:

$$(6) \quad A(t)\Phi = U(t) * (A_0(t)\Phi) * U(t)^{-1}.$$

Then the operator  $A(t)$  seems a reasonable candidate for the perturbation expansion of the RDE evolution operator at the time  $t$ . This operator  $A(t)$  seemingly can be symbolically denoted by conjugation by  $\exp tH^{renorm}/(ih)$ , where  $H^{renorm}$  is the renormalized (infinite) Hamiltonian.

## 3. LATER REMARK

It turned out that the key assumption of the paper about the possibility to renormalize the expression (5), equal to

$$(7) \quad T \exp \int_{t_0}^{t_1} \int g u(t, \mathbf{x})^4 / 4! dt d\mathbf{x}$$

in the Weyl algebra  $W_0$  (see [5] for notations), is false. While for the one-loop diagram with two vertices the renormalization can be performed, for the two-loop diagram with two vertices one meets problems. The author is grateful to I. V. Tyutin for pointing out this problem. This is related to the fact that the function

$$(8) \quad g(t, \mathbf{x}) = \begin{cases} g, & t_0 \leq t \leq t_1, \\ 0, & \text{otherwise} \end{cases}$$

on the space-time  $\mathbb{R}^4$  is not smooth. If instead of the integral (7) we take the integral

$$(9) \quad T \exp \int_{-\infty}^{\infty} \int g(t, \mathbf{x}) u(t, \mathbf{x})^4 / 4! dt d\mathbf{x}$$

for a smooth function  $g(t, \mathbf{x})$ , say, with compact support, then this expression seemingly can be renormalized in the Weyl algebra (which is similar to the Bogolyubov–Parasyuk theorem), and the corresponding operator in the Fock space gives the Bogolyubov  $S$ -matrix  $S(g)$  [8].

Actually, DE is well defined in perturbation theory in one-loop (quasiclassical) approximation, i. e. up to  $o(\hbar)$ . This is in accordance with results of Maslov and Shvedov [9], who defined complex germ in quantum field theory using the Bogolyubov  $S$ -matrix.

All these statements are discussed and proved in the book [10].

A slightly different definition of infinite dimensional Weyl algebra appeared in the paper [6]. For a finite dimensional theory of the Weyl algebra see, for example, [7], §18.5, and references therein.

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